Electromagnetic Corrections to Muon and Beta Decays when Mediated by a Vector Boson

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(Received 24 February 1964)

The radiative corrections to the weak decays have been calculated using two electromagnetic formalisms for the vector bosons. The energy spectrum for the electrons from muon decay is obtained and is found to reduce to the old form, obtained using the four fermion interaction, when the mass of the boson becomes infinite. The effective value of the Michel parameter is calculated. It seems unlikely that the electromagnetic effects of a boson would be observed through a study of the spectrum shape. The spectrum we have calculated satisfies Kinoshita's theorem. The results for muon decay are independent of the vector meson formalism employed. It is found that the bosons can reduce the corrections to the O^{14} decay lifetime to 1.2%, although the reduction depends upon the choice of formalism. The boson mass provides an effective cutoff which renders the over-all correction to the lifetime finite. This was not previously the case. Finally it is concluded that a vector meson can explain the muon lifetime discrepancy, but not if the current indications on its mass prove to be correct.

1. INTRODUCTION

THE discrepancy between the theoretical and experimental values of the muon's lifetime has been discussed many times. The most notable analysis of the then current situation was given by Feynman.[1] Since then, additional experiments have been performed which place the discrepancy as high as 4.6%; so that, in spite of the numerous uncertainties, there can be no doubt at all that the muon's lifetime is not well understood.

One of the strong candidtates for an explanation of this puzzle has always been the intermediate vector boson. Its popularity springs from the fact that for a suitably light boson, over half of the discrepancy is removed. This modification, however, decreases rapidly as the mass of the boson is increased. Experiments done at Brookhaven,[2] and recently at CERN[3] tend to confirm what is in fact assumed in this paper, namely that massive charged vector bosons mediate the strangeness-conserving weak interactions.

With this assumption, one is immediately forced to recalculate the electromagnetic corrections to muon and beta decays, since these are presently computed[4–6] using the four fermion interaction.[7] We may then consider what effect these new corrections will have upon the muon's lifetime.

In Sec. 2 we consider two formalisms for the description of the vector mesons, and derive the rules for Feynman diagrams to be used in each formalism. Section 3 is devoted to muon decay. We calculate the energy spectrum for the decay electron, with both radiative and nonradiative effects included, and compute the effect of the mesons upon the effective value of the Michel parameter. In order to know the effect of the meson on nuclear beta decay and hence upon the vector coupling constant, one must first calculate the corrections to neutron decay. This is done in Sec. 4. Section 5 employs the results obtained in 3 and 4 to consider afresh the lifetime of the muon.

2. RULES FOR FEYNMAN DIAGRAMS INVOLVING VECTOR BOSONS

(a) Proca Formalism

The vector boson field was first studied by Proca.[8] The free field Lagrangian is

$$\mathcal{L}^{\text{Proca}} = -\tfrac{1}{2} f_{\mu\nu}{}^* f^{\mu\nu} + M^2 \varphi_\mu \varphi^\mu ,$$

where φ_μ is the meson's field operator,

$$f_{\mu\nu} = \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu ,$$
$$\partial_\mu = \partial/\partial x^\mu ,$$
$$M = \text{mass of the boson}.$$

The free field commutation relations can be deduced,[9] and also the propagator. One has

$$\langle T(\varphi_\mu{}^*(x)\varphi_\nu(x'))\rangle_0 = -\left(g_{\mu\nu} + \frac{1}{M^2}\partial_\mu\partial_\nu\right)\tfrac{1}{2}\Delta_F(x-x') ,$$

which in momentum space becomes

$$\frac{-i}{(2\pi)^4}\frac{g_{\mu\nu}-(k_\mu k_\nu/M^2)}{k^2-M^2+i\epsilon} .$$

[1] R. P. Feynman and M. Gell-Mann, *Proceedings of the 1960 Annual International Conference on High Energy Physics at Rochester* (Interscience Publishers, Inc., New York 1960), pp. 501–508.

[2] G. Danby, J. M. Gaillard, K. Goulianos, L. M. Ledermann, N. Mistry, M. Schwartz, and J. Steinberger, Phys. Rev. Letters **9**, 36 (1962).

[3] J. M. Gaillard, New York meeting of American Phyiscal Society, Jan. 1964; Bull. Am. Phys. Soc. **9**, 40 (1964).

[4] R. E. Behrends, R. J. Finkelstein and A. Sirlin, Phys. Rev. **101**, 866 (1956).

[5] S. M. Berman, Phys. Rev. **112**, 267 (1959).

[6] T. Kinoshita and A. Sirlin, Phys. Rev. **113**, 1652 (1959).

[7] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

[8] A. Proca, J. Phys. Radium **7**, 347 (1936).

[9] G. Wentzel, *Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1949), p. 168.

The electromagnetic field a_μ is introduced minimally by by the substitution in $\mathfrak{L}$

$$\partial_\mu \to \partial_\mu - iea_\mu \text{ when acting on } \varphi_\mu,$$
$$\partial_\mu \to \partial_\mu + iea_\mu \text{ when acting on } \varphi_\mu^*.$$

Then the total Lagrangian is

$$\mathfrak{L} = \mathfrak{L}_{em} + \mathfrak{L}_M + \mathfrak{L}_I;$$

$\mathfrak{L}_I$ is the interaction Lagrangian

$$\mathfrak{L}_I = ie[(\partial_\mu\varphi_\nu^*)\varphi^\nu a^\mu - (\partial_\mu\varphi_\nu^*)\varphi^\mu a^\nu - a_\mu\varphi_\nu^*(\partial^\mu\varphi^\nu) + a_\mu\varphi_\nu^*(\partial^\nu\varphi^\mu)] - e^2 a_\mu\varphi_\nu^*\varphi^\nu a^\mu + e^2 a_\mu\varphi_\nu^*\varphi^\mu a^\nu.$$

The canonical formalism is employed and Matthews[10] is invoked to remove the normal dependent terms. We see finally that there are two vertices of the boson field with the electromagnetic field. The first involves the emission or absorption of one photon, and the second, the emission or absorption of two photons. The factors present in momentum space are given in Fig. 1.

The assumption that vector bosons mediate the weak processes means that the so-called "weak current" J_λ is *not* coupled to itself by a contact interaction,[7] but is instead coupled to the vector meson field φ_λ. The interaction Lagrangian is

$$\mathfrak{L}_I = fJ_\lambda\varphi^\lambda + fJ_\lambda^*\varphi^{*\lambda},$$

where

$$J_\lambda = \bar\psi_e\gamma_\lambda a\psi_{\nu_e} + \bar\psi_\mu\gamma_\lambda a\psi_{\nu_\mu},$$
$$a = \tfrac{1}{2}(1 - i\gamma_5),$$

ψ_e, ψ_{ν_e}, ψ_μ, ψ_{ν_μ} are the field operators corresponding to e^-, ν_e, μ^-, ν_μ, respectively. f is the coupling constant and is dimensionless.

Thus, there is just one type of vertex in this formalism. It involves a lepton, its neutrino, and the meson. At such a vertex there is a factor $-f\gamma_\lambda a$ in a Feynman diagram.

PROCA FORMALISM RULES

Element	Graph	Value
Internal boson line	μ —k→ ν	$\dfrac{-i}{(2\pi)^4}\dfrac{g_{\mu\nu} - \dfrac{k_\mu k_\nu}{M^2}}{k^2 - M^2 + i\epsilon}$
Lepton vertex	λ	$-f\gamma_\lambda a$
3-vertex	β, p′, p, α, μ	$e[p_\beta g_{\alpha\mu} + p'_\alpha g_{\beta\mu} - g_{\alpha\beta}(p+p')_\mu]$
4-vertex	β, α, ν, μ	$e^2[2g_{\mu\nu}g_{\alpha\beta} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}]$

FIG. 1. Proca formalism rules.

[10] P. T. Matthews, Phys. Rev. **76**, 684 (1949).

Element	Graph	Value
Internal A-line	A, μ —k→ ν	$\dfrac{-i}{(2\pi)^4}\dfrac{g_{\mu\nu}}{k^2 - M^2 + i\epsilon}$
Internal B-line	B, —k→	$\dfrac{i}{(2\pi)^4}\dfrac{1}{k^2 - M^2 + i\epsilon}$
Lepton-A vertex	A, λ	$-f\gamma_\lambda a$
Lepton-B vertex	B, k	$-f\dfrac{ik^\lambda}{M}\gamma_\lambda a$
Lepton-photon-B vertex	B, ν	$+\dfrac{ief}{M}\gamma_\lambda a$
Photon-A3 vertex	β, p′, A, p, A, α, μ	$-eg_{\alpha\beta}(p+p')_\mu$
Photon-B3 vertex	p′, B, p, B, μ	$+e(p+p')_\mu$
Photon-A4 vertex	β, A, A, α, ν, μ	$+e^2 2g_{\alpha\beta}g_{\mu\nu}$
Photon-B4 vertex	B, B, ν, μ	$e^2 2g_{\mu\nu}$

FIG. 2. Stückelberg formalism rules.

(b) Stückelberg Formalism

The Stückelberg formalism[11] replaces the field $\varphi_\mu(x)$ by two fields $A_\mu(x)$, $B(x)$

$$\varphi_\mu = A_\mu + (1/M)\partial_\mu B,$$

and the free field Lagrangian density is

$$\mathfrak{L}^{\text{Stück}} = -\partial_\mu A_\nu^*\partial^\mu A^\nu + \partial_\mu B^*\partial^\mu B + M^2 A_\mu^* A^\mu - M^2 B^* B.$$

The solutions must then be restricted by the imposed subsidiary condition

$$\partial_\mu\varphi^\mu(x) = 0,$$

or equivalently,

$$\partial_\mu A^\mu(x) - MB(x) = 0.$$

Then all the negative energies are contained in the B field, which turns out in effect to be uncoupled.[12] One can then derive the free field commutation relations and also the propagators. In momentum space the A field's propagator is

$$[-i/(2\pi)^4]g_{\mu\nu}/(k^2 - M^2 + i\epsilon),$$

and that of the B field is

$$[i/(2\pi)^4]1/(k^2 - M^2 + i\epsilon).$$

The electromagnetic field is introduced minimally, as

[11] E. Stückelberg, Helv. Phys. Acta **11**, 225, 299 (1938).
[12] W. E. Thirring, *Principles of Quantum Electrodynamics* (Academic Press Inc., London, 1958), p. 168.

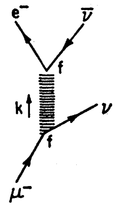

FIG. 3. Muon decay via a vector meson.

before, and we find that the interaction Lagrangian is

$$\mathcal{L}_I = ie[(\partial_\mu A_\nu^*)A^\nu a^\mu - A_\nu^*(\partial_\mu A^\nu)a^\mu] - e^2 A_\nu^* A^\nu a_\mu a^\mu + ie[B^*(\partial_\mu B)a^\mu - (\partial_\mu B^*)Ba^\mu] + e^2 B^* B a_\mu a^\mu.$$

Proceeding in the standard way via the canonical formalism and the interaction representation, we see finally that each of the two fields A_μ, B has two vertices with the electromagnetic field corresponding to the emission or absorption of one and two photons. The factors present in momentum space are given in Fig. 2.

Just as in the Proca formalism the weak current J_λ is coupled to the field φ_λ. In the Stückelberg formalism

$$J_\lambda\varphi^\lambda = J_\lambda A^\lambda + (1/M)J_\lambda\partial^\lambda B,$$

and when the electromagnetic field is introduced minimally, this becomes

$$J_\lambda A^\lambda + (1/M)J_\lambda(\partial^\lambda - iea^\lambda)B.$$

Thus we see that there are three types of vertices in which the leptons interact with the bosons. Two of these are simply the A or B being emitted or absorbed by the leptons. The third is a vertex peculiar to the B field. It involves the leptons, a photon, and the meson interacting at a point. This gauge invariant vertex is a direct consequence of the derivative coupling of the B mesons to the leptons, and is unique to this formalism. The rules for Feynman diagrams in momentum space are given in Fig. 2.

It has been observed[13] that there is no "normal" vector meson electrodynamical formalism. This is why we employ the two formalisms discussed above. That the formalisms we are using describe vector mesons with different electromagnetic properties is clear from the fact that their magnetic moments differ. The Proca boson has no anomalous magnetic moment, and consequently has a total moment of one Bohr magneton. Choice of the Stückelberg formalism implies that the boson has zero magnetic moment, or equivalently an anomalous moment of -1 magneton. Whether this is the only difference in the bosons so described is not presently clear. Of course, both formalisms yield the same matrix element for muon decay without electromagnetic corrections. This is clear because the electrodynamics of the meson do not then enter.

[13] J. A. Young, thesis, University of California, Lawrence Radiation Laboratory Report UCRL-9563, 1961 (unpublished).

3. ELECTROMAGNETIC CORRECTIONS TO MUON DECAY

We shall now use the formalisms discussed in the previous section to calculate the electromagnetic corrections to muon decay:

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu.$$

We assume that the boson is coupled to the leptons in the manner described. Then the diagram representing this process in the absence of electromagnetic effects is shown in Fig. 3. Eventually we expect the decay electron spectrum to consist of four parts: (i) The bare spectrum shape having Michel parameter $\rho = \frac{3}{4}$, which arises from the Fermi-type point interaction. (ii) Corrections due to nonlocal effects induced by the vector meson (but not including radiative effects). (iii) Radiative effects which are present for the four fermion interaction. (iv) Electromagnetic corrections arising solely from the presence of the charged vector boson propagating the process. These four classes may be characterized (for convenience):

(i) no bosons, no photons,
(ii) bosons, no photons,
(iii) photons, no bosons,
(iv) bosons and photons.

(i) and (ii) arise from the diagram shown in Fig. 3. (iii) has been written down previously.[4–6,14]

It is the fourth class which concerns us, although in the course of deriving it we shall inevitably write down the other terms. It is of interest to observe[15] that no two of the published results are exactly the same for the inner bremsstrahlung calculations. Our calculations agree with those of Kinoshita and Sirlin[6] for the contribution to class (iii). We expect that class (iv) will, in the main, contribute terms of the order of k^2/M^2 times the contribution from class (iii). k is the momentum carried by the boson in the bare process. For muon decay

$$k^2/M^2 \gtrsim m_\mu^2/M^2 < 5\times10^{-2},$$

where the upper limit is calculated by using the lower bound on M, namely, one kaon mass. Such terms will in general be negligible, and we therefore make the approximation (in the radiative corrections only) of neglecting terms of order k^2/M^2 *except* where they are multiplied by a large number, for example, like $\ln(m_e^2/M^2)$. Terms proportional to $m_e m_\mu/M^2$ are completely negligible.

As might be expected, a number of the Feynman graphs which we consider are divergent. These are handled in the standard way by introducing an ultra-

[14] V. P. Kuznetzov, Zh. Eksperim. i Teor. Fiz. **37**, 1102 (1959); **39**, 1722 (1960) [English transls.: Soviet Phys.—JETP **10**, 784 (1960); **12**, 1202 (1961)].

[15] C. R. Schumacher, Cornell University, thesis (unpublished).

violet cutoff λ. One replaces the photon propagator

$$\frac{1}{q^2}\to\frac{1}{q^2-\lambda_{\min}{}^2}\left(\frac{-\lambda^2}{q^2-\lambda^2}\right)^n.$$

$\lambda_{\min}$ is an infrared cutoff which prevents a low energy divergence, arising because the photon has zero mass. $\lambda_{\min}$ will disappear from the final answer because we consider both "pure" muon decay and the inner bremsstrahlung process

$$\mu^-\to e^-+\bar{\nu}_e+\nu_\mu+\gamma.$$

The integer n is chosen to be just large enough to secure convergence of *all* of the diagrams under consideration. It is important that the same integer n be used for all diagrams, since otherwise the procedure is not gauge invariant.[15]

When using a cutoff, the "rules of the game" are to let λ be infinite wherever possible. One retains only those terms which would be infinite in the limit of $\lambda\to\infty$. Similar considrations apply to $\lambda_{\min}$ and the limit $\lambda_{\min}\to 0$.

It turns out that when one calculates the total matrix element for muon decay, including electromagnetic corrections, there are many terms proportional to the bare matrix element M_0 which arises from Fig. 3. The total matrix element M has the form

$$M=M_0+\alpha(A+B(k))M_0+\alpha M_1,$$

where $\alpha=e^2/4\pi$, putting $\hbar=c=1$. A is a constant; $B(k)$ is a symbolic notation to indicate energy dependence. M_1 is a matrix element not equal to M_0. We see that M can be rewritten

$$M=(1+\alpha A)[M_0+\alpha B(k)M_0+\alpha M_1]$$

since we can neglect $O(\alpha^2)$. Thus the *constant* term A in no way affects the *shape* of the decay electron's spectrum, and could be ignored if we were only concerned with the spectrum's shape. Since we are ultimately interested in the muon's lifetime we must be more careful.

We can regard the factor $1+\alpha A$ as a coupling constant renormalization. The coupling constant for muon decay is derived from the decay of O^{14}. If an *identical* constant term αA arises in the calculations of the radiative corrections to O^{14} decay it too can be absorbed into the coupling constant and in this case will have no observable effect whatever since both coupling constants have undergone an identical renormalization. In this case the term αA can be dropped completely. However if the constant term αA arises in muon decay but *not* in nuclear beta decay then we may absorb αA into the coupling constant, as demonstrated, and it will not affect the spectrum shape, but we must retain it for use when calculating the lifetime of the muon. Examples of such a term would be one which depended on the mass of the muon, or a term whose sign differed when calculating beta decay.

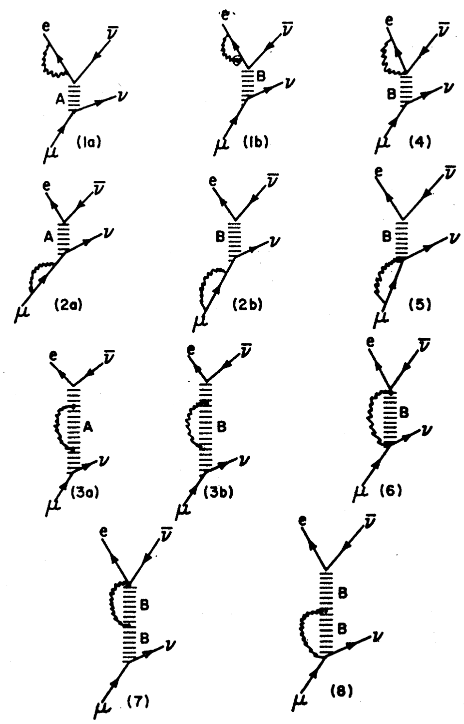

FIG. 4. Virtual photon diagrams: Stückelberg formalism.

Shaffer[16] has calculated the electromagnetic corrections to muon and nucleon decay using the Proca formalism. However he has neglected all boson effects in the bremsstrahlung calculation,[17] and consequently the spectrum he obtains does not satisfy Kinoshita's theorem.[18]

(a) Stückelberg Formalism

For historical reasons we calculate the radiative corrections first using the Stückelberg formalism. To order e^2, there are 19 virtual photon diagrams which contribute to muon decay. These are shown in Figs. 4 and 5. The labeling of these diagrams is self-explanatory. We need $n=2$ for this formalism.

The contributions of the self-energy diagrams (1), (2) (Fig. 4) can be written down immediately. Neglecting $m_e m_\mu/M^2$,

$$M_1+M_2=\theta\left(1+\frac{k^2}{M^2}\right)\left[\ln\frac{\lambda^2}{m_e m_\mu}-2\ln\frac{m_e m_\mu}{\lambda_{\min}{}^2}+\tfrac{7}{2}\right]\times(\bar{e}\gamma^\alpha a\nu_1)(\bar{\nu}_2\gamma_\alpha a\mu),$$

[16] R. A. Shaffer, Phys. Rev. 128, 1452 (1962).
[17] R. A. Shaffer (private communication).
[18] T. Kinoshita, J. Math Phys. 3, 650 (1962).

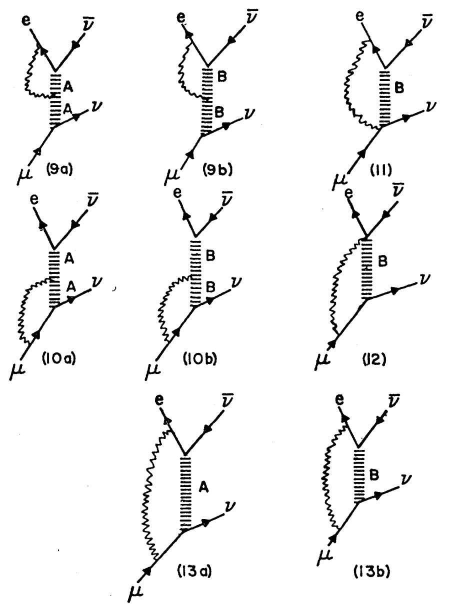

FIG. 5. Virtual photon diagrams: Stückelberg formalism.

where

$$\theta=\frac{\alpha i f^2}{2(2\pi)^3 M^2}\left(\frac{m_e m_\mu}{E_e E_\mu \omega_1 \omega_2}\right)^{1/2}\delta(p_\mu-p_e-p_{\bar{\nu}}-p_\nu).$$

E_e, E_μ, ω_1, ω_2 are the energies of e^-, μ^-, $\bar{\nu}_e$, ν_μ, respectively. p_e, p_μ, $p_{\bar{\nu}}$, p_ν are their four momenta. Without confusion we may use the same letter to denote the particle and its spinor.

After performing a mass renormalization, we find that M_3 is proportional to the bare matrix element M_0.

$$M_0=\frac{\beta}{1-(k^2/M^2)}(\bar{e}\gamma^\alpha a\nu_1)(\bar{\nu}_2\gamma_\alpha a\mu),$$

where

$$\beta=-4\pi\theta/\alpha.$$

Consequently, we may by means of a coupling constant renormalization set $M_3=0$. Both M_4 and M_5 are proportional to $m_e m_\mu/M^2$, and so can be neglected: $M_4=M_5=0$. After coupling constant renormalization

$$M_6=\theta(k^2/M^2)\ln(\lambda^2/M^2)(\bar{e}\gamma^\alpha a\nu_1)(\bar{\nu}_2\gamma_\alpha a\mu).$$

This term must be retained as we have not so far specified the size of the cutoff. Use of the Dirac equation renders both M_7, M_8 of order $m_e m_\mu/M^2$, and thus to our approximation $M_7=M_8=0$. Use of the standard reduction methods yields

$$M_{9_a}=-\theta(2p_e\cdot k/M^2)\ln(m_e^2/M^2)(\bar{e}\gamma^\alpha a\nu_1)(\bar{\nu}_2\gamma_\alpha a\mu)$$

after renormalizing the coupling constant. Similarly

$$M_{10_a}=-\theta(1+k^2/M^2)[\ln(\lambda^2/M^2)-1](\bar{e}\gamma^\alpha a\nu_1)(\bar{\nu}_2\gamma_\alpha a\mu).$$

For this diagram however we may *not* renormalize the coupling constant since it is possible that the sign will differ when we calculate beta decay. Use of the Dirac equation leads us to $M_{9_b}=0$. Similarly, $M_{10_b}=0$. After coupling constant renormalization,

$$M_{11}=-\theta(k^2/M^2)\ln(\lambda^2/M^2)(\bar{e}\gamma^\alpha a\nu_1)(\bar{\nu}_2\gamma_\alpha a\mu).$$

Similar considerations as were applied to M_{10_a} lead us to

$$M_{12}=\theta[\ln(\lambda^2/M^2)-1](\bar{e}\gamma^\alpha a\nu_1)(\bar{\nu}\gamma_\alpha a\mu).$$

The two remaining diagrams (13a), (13b) are an order of magnitude more difficult to calculate than those so far considered. A four-sided loop integration has been done exactly by Wu[19] for the case when all four sides of the loop are scalar mesons. His answer involves 192 Spence functions.[20] This function, written $L(x)$, is defined by

$$L(x)=\int_0^x dt\frac{\ln(1-t)}{t}=-\sum_{n=1}^{\infty}\frac{t^n}{n^2}.$$

Fortunately, because of the approximations we can make, our answer is somewhat simpler. The integration can be done by an iterative method.[21] Combining both contributions,

$$\begin{aligned}M_{13}=\theta\Bigg\{\Bigg[&\frac{2p_e\cdot p_\mu}{M^2}\ln\frac{m_e^2}{M^2}+\left(1+\frac{k^2}{M^2}\right)T+4\left(1+\frac{k^2}{M^2}-\frac{p_e\cdot p_{\bar{\nu}}}{M^2}\right)\ln\frac{m_e m_\mu}{2p_e\cdot p_\mu}\\&-\frac{2p_e\cdot p_\mu}{q^2}\ln\frac{2p_e\cdot p_\mu}{M^2}+\frac{m_\mu^2}{q^2}\ln\frac{m_\mu^2}{M^2}-\tfrac{1}{2}-\ln\frac{\lambda^2}{M^2}+2\ln\frac{2p_e\cdot p_\mu}{m_\mu^2}\Bigg](\bar{e}\gamma^\alpha a\nu_1)(\bar{\nu}_2\gamma_\alpha a\mu)\\&-\frac{2m_\mu}{q^2}\ln\frac{2p_e\cdot p_\mu}{m_\mu^2}(\bar{e}\mathbf{q}a\nu_1)(\bar{\nu}_2 a'\mu)+\frac{2}{q^2}\left[1+\frac{4p_e\cdot p_\mu-m_\mu^2}{q^2}\ln\frac{2p_e\cdot p_\mu}{m_\mu^2}\right](\bar{e}\mathbf{q}a\nu_1)(\bar{\nu}_2\mathbf{q}a\mu)\Bigg\},\end{aligned}$$

where $T=2(\omega+\ln x)(\omega+\ln x-2\omega_<)-2\ln x\ln(1-x)+2L(x)-2L(1)$, $q=p_e-p_\mu$, $a'=\tfrac{1}{2}(1+i\gamma_5)$ $\omega=\ln(m_\mu/m_e)$,

[19] A. C–T. Wu, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **33**, No. 3 (1961).
[20] W. Spence, *An Essay on Logarithmic Transcendents* (John Murray, London and Archibald Constable and Company, Edinburgh, 1809).
[21] D. Bailin, thesis, Cambridge University Library (unpublished).

$\omega_< = \ln(\lambda_{\min}/m_e)$, and $x=1-q^2/m_\mu^2$. The variable x is the decay electron's energy measured in units of its maximum possible value: $\frac{1}{2}m_\mu$.

We can now sum the contributions to evaluate the virtual photon matrix element ΔM_V.

$$\Delta M_V=\sum_{i=1}^{13} M_i$$

$$=\theta\left\{\left(1+\frac{k^2}{M^2}\right)\left[3(1-\omega)+2(\omega+\ln x-1)(\omega+\ln x-2\omega_<)-\frac{x}{1-x}\ln x-2\ln x\ln(1-x)+2L(x)-2L(1)\right]\right.$$

$$\left.\times(\bar{e}\gamma^\alpha a\nu_1)(\bar{\nu}_2\gamma_\alpha a\mu)-\frac{2m_\mu}{q^2}\ln\frac{2p_e\cdot p_\mu}{m_\mu^2}(\bar{e}\boldsymbol{q}a\nu_1)(\bar{\nu}_2 a'\mu)+\frac{2}{q^2}\left(1+\frac{4p_e\cdot p_\mu-m_\mu^2}{q^2}\ln\frac{2p_e\cdot p_\mu}{m_\mu^2}\right)(\bar{e}\boldsymbol{q}a\nu_1)(\bar{\nu}_2\boldsymbol{q}a\mu)\right\}.$$

We see that this result is just that of Behrends *et al.*[4] multiplied by the shape-dependent factor $[1+(k^2/M^2)]$. It is of interest to note that just as in the four fermion case the cutoff dependence has canceled. This is not especially surprising though, since all we are saying is that with the *particular* coupling constant renormalization we have used we can make the muon decay result cutoff independent. It will be seen that we have retained terms from diagrams (1), (2) (Fig. 4), which could have been omitted as a coupling constant renormalization. This was done simply for ease of comparison with Behrends *et al.*[4] So long as we perform an identical coupling constant renormalization for the nucleon case, it makes no difference. We must now compute the transition probability $|M_0+\Delta M_V|^2$. After performing spin summations, and integrating over the phase space of the neutrino and antineutrino, we obtain for the virtual photon contribution to the transition probability

$$\begin{aligned}P_V(x)dx=&[f^4m_\mu^5/96(2\pi)^3M^4]dx\{[x^2(3-2x)\\&-cx^3(x-2)+\tfrac{3}{10}c^2x^4(5-2x)]-(\alpha/2\pi)\\&\times[\{(\omega+\ln x-1)(\omega+\ln x-2\omega_<)+3(1-\omega)\\&+1-\ln x-2\ln x\ln(1-x)+2L(x)-2L(1)\}\\&\times[x^2(3-2x)-cx^3(x-2)]+3x^2\ln x]\}\,,\end{aligned}$$

where $c=m_\mu^2/M^2$.

(b) Proca Formalism

There are six virtual photon diagrams which contribute to muon decay in the Proca theory. These are shown in Fig. 6. We must now use a cutoff cubed ($n=3$) since this is required to secure convergence of diagram (16). We have, therefore,

$$\begin{aligned}M_{14}+M_{15}=&\theta[1+(k^2/M^2)][\ln(\lambda^2/m_em_\mu)\\&-2\ln(m_em_\mu/\lambda_{\min}^2)+3](\bar{e}\gamma^\alpha a\nu_1)(\bar{\nu}_2\gamma_\alpha a\mu)\,.\end{aligned}$$

Diagram (16) is in fact quadratically divergent, but after mass and charge renormalizations we are left with

$$M_{16}=\theta(k^2/6M^2)\ln(\lambda^2/M^2)(\bar{e}\gamma^\alpha a\nu_1)(\bar{\nu}_2\gamma_\alpha a\mu)\,.$$

After coupling constant renormalization we have

$$M_{17}=-\theta\left[\frac{2p_e\cdot k}{M^2}\ln\frac{m_e^2}{M^2}+\frac{k^2-p_e\cdot k}{M^2}\ln\frac{\lambda^2}{M^2}\right]\times(\bar{e}\gamma^\alpha a\nu_1)(\nu_2\gamma_\alpha a\mu)\,.$$

Similarly,

$$\begin{aligned}M_{18}=&-\theta[\ln(\lambda^2/M^2)-\tfrac{3}{2}]\\&\times\{[\tfrac{3}{2}[1+(k^2/M^2)]+(k^2-p_\mu\cdot k)/M^2]\\&\times(\bar{e}\gamma^\alpha a\nu_1)(\bar{\nu}_2\gamma_\alpha a\mu)+(1/M^2)(\bar{e}p_\mu a\nu_1)(\bar{\nu}_2\boldsymbol{k}a\mu)\}\,.\end{aligned}$$

As before, we may not perform coupling constant renormalization in this case. Since the electrodynamics of the boson do not enter into diagram (19) one might expect its contribution to equal that of diagram (13) in the Stückelberg formalism. However, this is not quite true since we are using different powers of the cutoff in the two cases. To obtain M_{19} from M_{13} one simply replaces $\ln(\lambda^2/M^2)$ by $\ln(\lambda^2/M^2)-\frac{1}{2}$. Thus, summing for

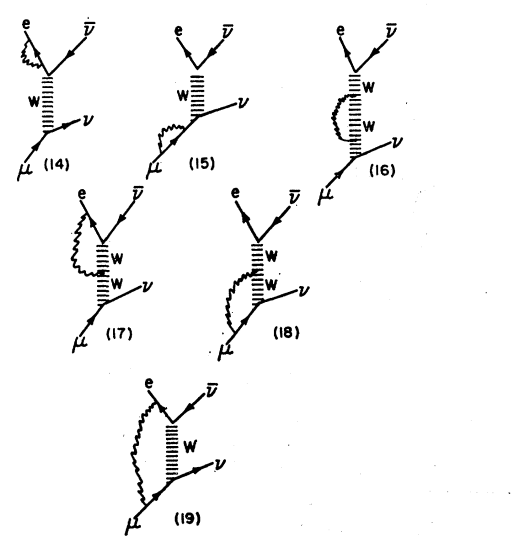

FIG. 6. Virtual photon diagrams: Proca theory.

the virtual photon matrix element in this formalism

$$\Delta M_V{}^{\text{Proca}}=\sum_{i=14}^{19} M_i$$
$$=\Delta M_V{}^{\text{Stück.}}+\theta\left\{\left[\frac{k^2}{6M^2}-\frac{k^2-p_e\cdot k}{M^2}-\frac{k^2-p_\mu\cdot k}{M^2}-\frac{3}{2}\left(1+\frac{k^2}{M^2}\right)\right](\bar{e}\gamma^\alpha a\nu_1)(\bar{\nu}_2\gamma_\alpha a\mu)\right.$$
$$\left.-\frac{1}{M^2}(\bar{e}\boldsymbol{p}_\mu a\nu_1)(\bar{\nu}_2\boldsymbol{k}a\mu)\right\}\left(\ln\frac{\lambda^2}{M^2}-\frac{3}{2}\right).$$

For reasonable values of λ, M we may neglect $O([k^2/M^2]\ln[\lambda^2/M^2])$ and we have then

$$\Delta M_V{}^{\text{Proca}}-\Delta M_V{}^{\text{Stück.}}=\theta\tfrac{3}{2}[\tfrac{3}{2}-\ln(\lambda^2/M^2)](\bar{e}\gamma^\alpha a\nu_1)(\bar{\nu}_2\gamma_\alpha a\mu).$$

The difference between these two matrix elements corresponds simply to a coupling constant renormalization. Thus as far as the *shape* of the virtual photon contribution to the spectrum is concerned the two formalisms yield identical results. Ultimately, when we consider the lifetime of the muon it is possible that this term will make some distinction between the two formalisms. For the moment, $P_V(x)$ which we have written down previously represents the virtual photon spectrum shape in either formalism.

(c) Bremsstrahlung Diagrams

We must now consider the inner bremsstrahlung process[22]

$$\mu^-\to e^-+\bar{\nu}_e+\nu_\mu+\gamma.$$

This is necessary in order to remove the infrared divergence which is present in the virtual photon spectrum. In the Stückelberg formalism there are eight diagrams which contribute. These are shown in Fig. 7. The photon's four momentum is K, and its polarization vector is $\epsilon^{(\lambda)}$; the λ is usually left understood.

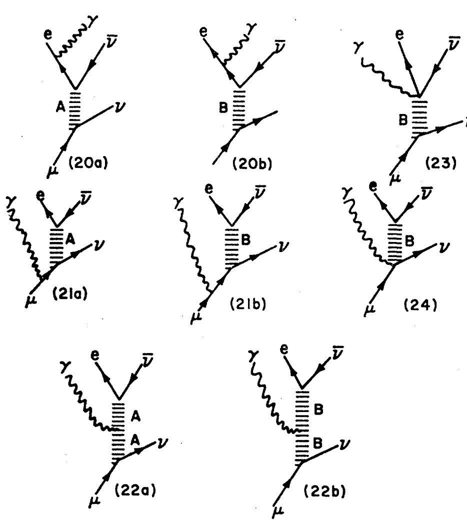

FIG. 7. Bremmstrahlung diagrams in Stückelberg theory.

[22] The possibility of directly observing vector meson effects by a study of this process is the subject of a separate paper. D. Bailin, Nuovo Cimento (to be published). R. C. Brunet, Nuovo Cimento **30**, 1317 (1963).

We obtain

$$M_{20a}=\varphi\left(\bar{e}\boldsymbol{\varepsilon}\frac{1}{\boldsymbol{p}_e+\boldsymbol{K}-m_e}\gamma^\alpha a\nu_1\right)(\bar{\nu}_2\gamma_\alpha a\mu)(k_1{}^2-M^2)^{-1},$$

where

$$\varphi=\frac{ief^2}{(2\pi)^{7/2}}\left(\frac{m_e m_\mu}{E_e E_\mu\omega_1\omega_2 2K_0}\right)^{1/2}\times\delta(p_\mu-p_e-p_\nu-p_{\bar{\nu}}-K),\quad k_1=p_\mu-p_\nu.$$

Also

$$M_{20b}=-(\varphi/M^2)(\bar{e}\boldsymbol{\varepsilon}[1/(\boldsymbol{p}_e+\boldsymbol{K}-m_e)](\boldsymbol{p}_e+\boldsymbol{K})a\nu_1)\times(\bar{\nu}_2\boldsymbol{k}_1 a\mu)(k_1{}^2-M^2)^{-1},$$
$$M_{21a}=\varphi(\bar{e}\gamma^\alpha a\nu_1)\times(\bar{\nu}_2\gamma_\alpha a[1/(\boldsymbol{p}_\mu-\boldsymbol{K}-m_\mu)]\boldsymbol{\varepsilon}\mu)(k_2{}^2-M^2)^{-1},$$

where

$$k_2=p_e+p_{\bar{\nu}}.$$

Use of Dirac's equation shows that to our approximation

$$M_{21b}=0,$$
$$M_{22a}=\varphi(\bar{e}\gamma^\alpha a\nu_1)(\bar{\nu}_2\gamma_\alpha a\mu)(k_1{}^2-M^2)^{-1}\times(k_2{}^2-M^2)^{-1}\epsilon\cdot(k_1+k_2),$$

and

$$M_{22b}=0.$$

Also

$$M_{23}=(\varphi/M^2)(\bar{e}\boldsymbol{\varepsilon}a\nu_1)(\bar{\nu}_2\boldsymbol{k}_1 a\mu)(k_1{}^2-M^2)^{-1},$$
$$M_{24}=0.$$

The total bremsstrahlung matrix element is then

$$M_{IB}=\sum_{i=20}^{24}M_i,$$

and the transition probability $|M_{IB}|^2$ is what we must calculate.

Since we are working only to order k^2/M^2, we have

$$|M_{IB}|^2=|M_{20a}+M_{21a}|^2+2\,\text{Re}(M_{20a}{}^\dagger+M_{21a}{}^\dagger)(M_{20b}+M_{22a}+M_{23}).$$

The transition probability is computed by averaging over initial spin states and summing over final spin states. Included in the latter is a sum over the polarization vectors of the photon. To handle the infrared problem consistently we must give the photon a small mass $\lambda_{\min}$.[5] We must therefore sum over the *three* polarization vectors possessed by a vector meson with nonzero mass. The summation takes the form

$$\sum_{\lambda=1}^{3} \epsilon_\mu^{(\lambda)}\epsilon_\nu^{(\lambda)} = -g_{\mu\nu} + K_\mu K_\nu/\lambda_{\min}^2 .$$

The $\epsilon_\mu^{(\lambda)}$ satisfy

$$\epsilon\cdot K = 0 \quad \text{and} \quad \epsilon^2 = -1 .$$

In our case the matrix element has the form

$$M_{IB} = M\cdot\epsilon^{(\lambda)} ,$$

so that

$$\sum_\lambda |M_{IB}|^2 = -|M|^2 + (1/\lambda_{\min}^2)|M\cdot K|^2 .$$

However,

$$M\cdot K = 0 .$$

This is expected on grounds of gauge invariance for a zero-massed photon, but is, in fact, exactly true[21] even when $\lambda_{\min}\neq 0$. It can also be shown to be true for the case of the four fermion interaction. So we have that

$$\sum_\lambda |M_{IB}|^2 = -|M|^2 .$$

After the remaining spin sums have been performed we must integrate over the phase space of the neutrino and antineutrino, and also that of the photon. We obtain finally for the contribution from the bremsstrahlung process in this formalism

$$\begin{aligned} P_{IB}(x)dx &= [\alpha f^4 m_\mu^5/(2\pi)^4 M^4 96]dx\{[(3-2x)x^2 - cx^3(x-2)] \\ &\quad\times[2(\omega+\ln x-1)(2\ln(1-x)-\ln x+\omega-2\omega_<) \\ &\quad -2L(x)+2L(1)-(2(1-x)/x)\ln(1-x)] \\ &\quad +(5/3)(1-x)^2+\tfrac{1}{3}(\ln x+\omega-1)(1-x) \\ &\quad\times[5+17x-34x^2+\tfrac{1}{2}c(5+13x+49x^2-43x^3)]\} . \end{aligned}$$

It should be noted that if one puts $c=(m_\mu/M)^2=0$, one does not quite recover the expression given by Berman.[5]

In the Proca formalism there are just three diagrams which contribute to the bremsstrahlung process. These are shown in Fig. 8. Since the electrodynamics of the boson do not enter into diagrams (25), (26), we have

$$M_{25} = M_{20a} + M_{20b} ,$$
$$M_{26} = M_{21a} + M_{21b} .$$

To our approximation we have

$$M_{27} = (\varphi/M^4)(\bar{e}\gamma^\alpha a\nu_1)(\bar{\nu}_2\gamma^\beta a\mu) \times[\epsilon\cdot(k_1+k_2)g_{\alpha\beta} - k_{2\beta}\epsilon_\alpha - k_{1\alpha}\epsilon_\beta] .$$

It turns out that the additional terms in this formalism do not supply an appreciable contribution in our approximation, and consequently the expression given

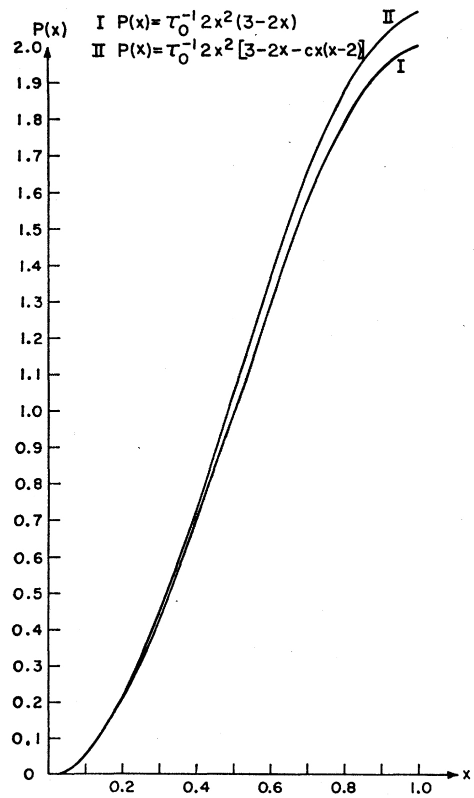

FIG. 9. Corrections to bare spectrum resulting from vector mesons.

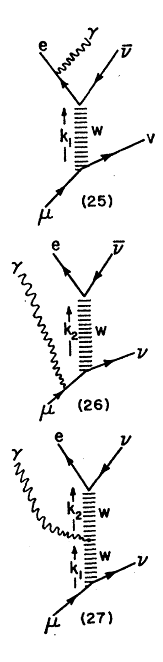

FIG. 8. Bremmstrahlung diagrams in Proca theory.

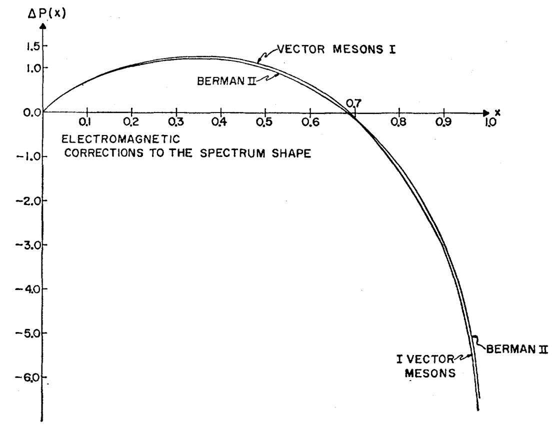

FIG. 10. Electromagnetic corrections to the spectrum shape.

previously represents the bremsstrahlung contribution to the transition probability in either formalism.

So the net result is that the shape of the decay electron's spectrum is independent of the vector boson formalism employed. Combining the contributions of the virtual photons and the inner bremsstrahlung process, we obtain the total corrected transition probability:

$$P(x)dx=[f^4m_\mu{}^5/96(2\pi)^3M^4]x^2dx[3-2x-cx(x-2) +\tfrac{3}{10}c^2x^2(5-2x)+(\alpha/2\pi)f(x)],$$

where

$$f(x)=2[3-2x-cx(x-2)]R(x)+(6-6x)\ln x +[(1-x)/3x^2][(\omega+\ln x)(5+17x-34x^2 +\tfrac{1}{2}c[5+13x+49x^2-43x^3])-22x+34x^2],$$

$$R(x)=-2L(x)+2L(1)-2+\omega\left(\tfrac{3}{2}+2\ln\frac{1-x}{x}\right) -\ln x(2\ln x-1)+[3\ln x-1-(1/x)]\ln(1-x).$$

We are using the notation of Ref. 6. The infrared divergence has disappeared, of course; as it stands the spectrum diverges at $x=1$. The correct procedure is to replace $\ln(1-x)$ by $\ln[1-x+(\Delta x/e)]$, where Δx is the experimental energy resolution measured in these units. The change in the bare spectrum due to the presence of vector bosons is shown in Fig. 9. Figure 10 shows the effect of the bosons on the electromagnetic corrections.

We may calculate the effective value of the Michel parameter for the spectrum. Using a least-squares fit in the range $0\leq x\leq 0.95$, we obtain, after some labor

$$\rho_{\text{eff}}=\tfrac{3}{4}+0.3387c+0.2532c^2-(\alpha/2\pi)[47.99-8.165c].$$

For the largest possible value of $c=0.0458$,

$$\rho_{\text{eff}}=\tfrac{3}{4}-0.0393.$$

It is unlikely that even the most sophisticated experiments currently possible could detect the electromagnetic boson effects.

We may integrate the spectrum over all values of x to obtain the lifetime.

$$\tau_\mu{}^{-1}=\tau_0{}^{-1}\left[1+\tfrac{3}{5}c+\tfrac{2}{5}c^2+\frac{\alpha}{2\pi}\left(\frac{25}{4}-\pi^2\right)(1+\tfrac{3}{5}c)\right],$$

$$\tau_\mu=\tau_0\left(1-\tfrac{3}{5}c-\frac{c^2}{25}\right)\left[1-\frac{\alpha}{2\pi}\left(\frac{25}{4}-\pi^2\right)\right].$$

τ_0 is the lifetime calculated in the absence of electromagnetic corrections. It involves of course the coupling constant which is at present not determined. Note that the *lifetime* is unaltered if one lets the mass of the electron become zero. Although the spectrum depends logarithmically upon m_e (through ω) the dependence on ω is removed when one integrates over x. Actually this is just a verification of a general theorem of Kinoshita.[18] It does, however, provide a useful and independent check on the spectrum we have obtained.

Now,

$$\tfrac{3}{5}c+c^2/25<2.76\times10^{-2}.$$

Consequently, $-\Delta\tau/\tau_0<2.34\%$ when one includes both radiative and nonradiative corrections. The change in the lifetime which we are calculating here is, of course, the correction arising from the change in *shape* of the decay electron's spectrum. Corrections due to the coupling constant cannot be considered until we have dealt with beta decay. Consequently the numbers we have given are the same for both formalisms.

4. ELECTROMAGNETIC CORRECTIONS TO NEUTRON DECAY AND NUCLEAR BETA DECAY

We have seen in Sec. 3 that the effect of the bosons on the decay electron's spectrum shape cannot explain all of the muon's lifetime discrepancy. Indeed the electromagnetic effects have little or no effect upon the lifetime. What effect there is is due to the nonradiative correction to the bare spectrum induced by nonlocality.

However, it is still conceivable that the bosons can explain the discrepancy. As was remarked before, in order to calculate τ_0, the bare lifetime, we must first know the coupling constant f, or more precisely f^2. This is assumed to be the same as the vector coupling constant for nuclear beta decay. If the bosons have a significant effect upon the beta decay calculation, then this will affect f which is calculated from the *observed* beta decay lifetime. And consequently the theoretical value of the *muon* lifetime will be changed because of these corrections to the coupling constant. Actually, the process used is the $0^+\to0^+$ decay of O^{14}.

To calculate the corrections to nuclear beta decay one treats the nucleus as a collection of independent nucleons. Corrections due to nuclear physics can then be estimated afterwards.[23,24] We must therefore consider what effects the bosons will have upon the decay of a single nucleon. This is the problem to which we now turn.

We assume that the interaction of the mesons with the nucleons is given by

$$\mathfrak{L}_I = j_\alpha \varphi^\alpha + j_\alpha{}^* \varphi^{\alpha *},$$

where

$$j_\alpha = f\bar{\psi}_n \gamma_\alpha a'' \psi_p, \quad a'' = \tfrac{1}{2}(1 - \mathrm{Li}\gamma_5).$$

ψ_n, ψ_p are the field operators for the neutron and proton, respectively. The rules for Feynman diagrams which follow from this interaction are completely analogous to those already written down for the lepton currents. So we will not repeat them. The number L is about 1.25 for neutron decay. The Feynman diagram representing the decay of the neutron, in the absence of electromagnetic effects, is shown in Fig. 11.

$$n \rightarrow p + e^- + \bar{\nu}_e.$$

The calculations for this process are simpler, in one respect only, than those for muon decay. Because of the small difference of the neutron and proton masses we may neglect *all* terms of order k^2/M^2, where $k = p_e + p_{\bar{\nu}}$, as before,

$$k^2/M^2 = O(m_e^2/M^2) \gtrsim 10^{-6}.$$

Thus the bare matrix element is

$$M_0 = \frac{-if^2}{M^2 4\pi^2}\left(\frac{m_e M_n M_p}{E_e E_n E_p \omega_1}\right)^{1/2} \times (\bar{e}\gamma^\alpha a\nu_1)(\bar{p}\gamma_\alpha a'' n)\delta(N - P - p_e - p_{\bar{\nu}}).$$

N, P are four momenta of neutron and proton. E_n, E_p, and M_n, M_p are their energies and masses, respectively.

Complication arises because in this process we may not take the boson's mass M as being large compared to those of the other particles involved. Indeed M may well be less than M_p. Fortunately many of the diagrams we are led to consider differ only in that they have a nucleon current replacing the muon current. The matrix elements of such diagrams can be written down at once.

(a) Stückelberg Formalism

For the Stückelberg formalism there are nineteen virtual photon diagrams which contribute to neutron decay. They are shown in Figs. 12 and 13. They are completely analogous to those arising in muon decay and we have labeled the diagrams accordingly. It should be noted that in this process we have both charged particles in the final state, whereas in muon decay this was not the case. This difference occasionally has an important effect, as we shall see.

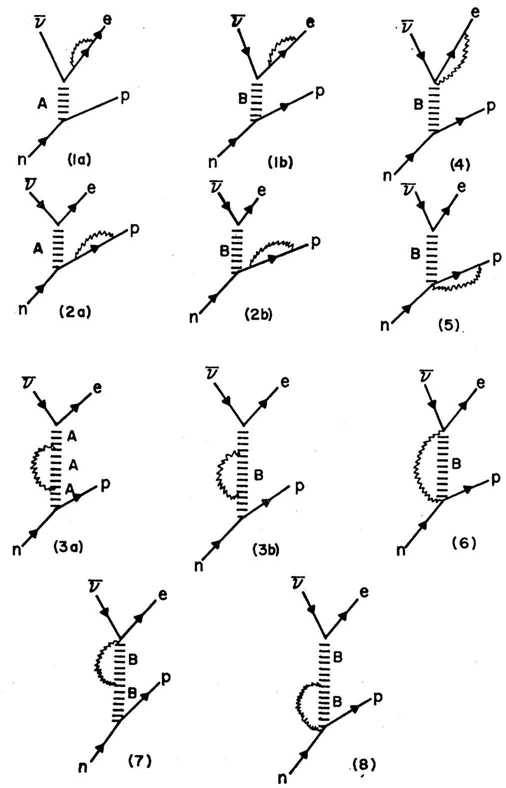

FIG. 12. Virtual photon diagrams: Stückelberg formalism.

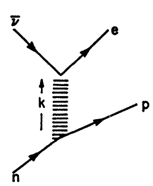

FIG. 11. Neutron decay via a vector meson.

By a simple substitution

$$M_1 + M_2 = \theta[\ln(\lambda^2/m_e M_p) - 2\ln(m_e M_p/\lambda_{\min}) + \tfrac{7}{2}] \times (\bar{e}\gamma^\alpha a\nu_1)(\bar{p}\gamma_\alpha a'' n),$$

where now

$$\theta = \frac{\alpha i f^2}{2(2\pi)^3 M^2}\left(\frac{m_e M_p M_n}{E_e E_n E_p \omega_1}\right)^{1/2} \delta(N - P - p_e - p_{\bar{\nu}}).$$

Using the same coupling constant renormalization as was used for muon decay $M_3 = 0$. Just as for muon decay, $M_4 = M_5 = 0$. Coupling constant renormalization and neglect of k^2/M^2 imply, $M_6 = 0$, $M_7 = M_8 = 0$, $M_{9a} = M_{9b} = 0$.

Diagram (10) (Fig. 5) is the first one which differs essentially from its counterpart in muon decay. Our previous approximation is no longer valid. We may, however, set $k = 0$ without neglecting significant terms. By virtue of a freak cancellation we are left finally with

$$M_{10a} = -\theta[\ln(\lambda^2/M^2) - 1](\bar{e}\gamma^\alpha a\nu_1)(\bar{p}\gamma_\alpha a'' n).$$

[23] W. M. MacDonald, Phys. Rev. **110**, 1420 (1958).
[24] H. A. Weidenmuller, Phys. Rev. **127**, 537 (1962).

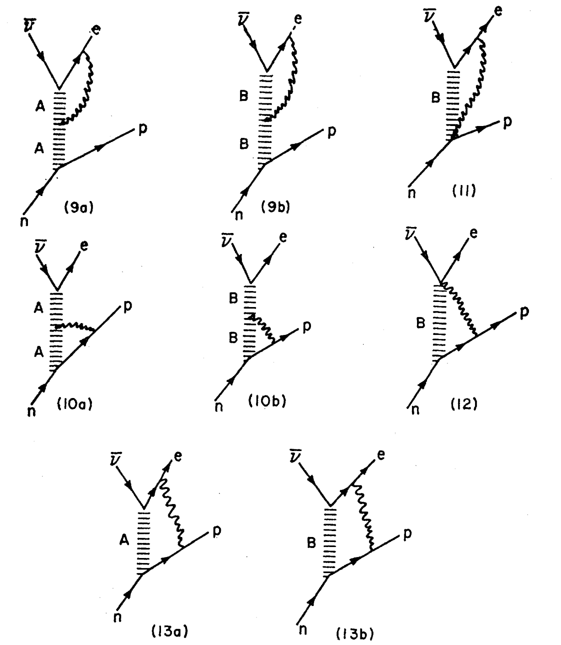

FIG. 13. Virtual photon diagrams: Stückelberg formalism.

Dirac's equation allows us to set $M_{10b}=0$. Coupling constant renormalization implies $M_{11}=0$. Diagram (12) (Fig. 5), like (10), must be recalculated. Setting $k=0$ we obtain finally

$$M_{12}=\theta\{[\ln(\lambda^2/M^2)+(1-\tfrac{1}{2}\omega)\ln\omega-(4\omega-\omega^2)^{1/2}t(\omega)]\times(\bar{e}\gamma^\alpha a\nu_1)(\bar{p}\gamma_\alpha a''n)+[-1+(\tfrac{1}{2}\omega-1)\ln\omega+(4\omega-\omega^2)^{1/2}t(\omega)][\bar{e}(\boldsymbol{P}/M_p)a\nu_1](\bar{p}a''n),$$

where

$$\omega=M^2/M_p{}^2>0.277,$$

and

$$t(\omega)=\tan^{-1}\frac{2-\omega}{(4\omega-\omega^2)^{1/2}}-\tan^{-1}\frac{-\omega}{(4\omega-\omega^2)^{1/2}}.$$

The form we have written assumes that $\omega<4$. Should it turn out that $\omega>4$ we use the correct logarithmic continuation of the inverse tangent. That is

$$\frac{2}{(4\omega-\omega^2)^{1/2}}t(\omega)\rightarrow\frac{1}{(\omega^2-4\omega)^{1/2}}\ln\frac{\omega+(\omega^2-4\omega)^{1/2}}{\omega-(\omega^2-4\omega)^{1/2}}.$$

As before, diagram (13) (Fig. 5) is the most difficult case to handle. Since the invariant center-of-mass energy of the electron and proton is above the normal threshold some of the integrals we encounter have singularities in the range of integration. This means that the amplitude for the process is complex.

Actually, we may throw away the imaginary part of the integral because it never contributes to the transition probability.[21] Consequently, we will only write down the *real* parts of the integrals in all that follows. For simplicity in doing the integrals we have assumed that $L=1$, or $a''=a$. Although it is possible that the pseudovector coupling can induce a contribution to the vector coupling by means of the electromagnetic effects, there is good reason to believe that this assumption does not, in fact, affect the results appreciably.[21]

With this assumption

$$M_{13a}=-M^2\theta\{[4p_e\cdot PJ_{31}-4J_{33}+8J_{21}]\times(\bar{e}\gamma^\alpha a\nu_1)(\bar{p}\gamma_\alpha an)+2(\bar{e}\gamma^\alpha a\nu_1)(\bar{p}p_eJ_{32}\gamma_\alpha an)-2(\bar{e}\boldsymbol{P}J_{32}\gamma^\alpha a\nu_1)(\bar{p}\gamma_\alpha an)\},$$

$$M_{13b}=\theta\{[1+6J_1+6J_{23}-J_{35}](\bar{e}\gamma^\alpha a\nu_1)(\bar{p}\gamma_\alpha an)+6(\bar{e}\boldsymbol{P}a\nu_1)(\bar{p}J_{22}an)-2(\bar{e}\boldsymbol{P}a\nu_1)(\bar{p}J_{34}an)\},$$

where

$$6J_1=\ln\frac{M^2}{\lambda^2}-\frac{11}{6}+\frac{17}{20}\omega-\frac{\omega^2}{10}+\left(\frac{\omega^3}{20}-\frac{\omega^2}{2}+\frac{3\omega}{2}-1\right)\ln\omega+\left(-\frac{\omega^4}{20}+\frac{3\omega^3}{5}-\frac{12}{5}\omega^2+\frac{16}{5}\omega\right)\frac{2}{(4\omega-\omega^2)^{1/2}}t(\omega),$$

$$J_{21}=\frac{1}{M_p{}^2}\left\{\frac{1}{6}+\left(-\frac{\omega}{12}+\frac{1}{2}\right)\ln\omega+\left(\frac{\omega^2}{12}-\frac{2\omega}{3}+\frac{4}{3}\right)\frac{2t(\omega)}{(4\omega-\omega^2)^{1/2}}\right\},$$

$$J_{22}=\frac{P}{M_p{}^2}\left\{\frac{3}{8}-\frac{\omega}{12}+\left(\frac{1}{\omega}-\frac{\omega}{4}+\frac{\omega^2}{24}\right)\ln\omega+\left(-\frac{4}{3}\omega+\frac{2\omega^2}{3}-\frac{\omega^3}{12}\right)\frac{t(\omega)}{(4\omega-\omega^2)^{1/2}}\right\},$$

$$J_{23}=\frac{1}{6}-\frac{31}{120}\omega+\frac{\omega^2}{20}-\left(\frac{\omega^3}{40}-\frac{\omega^2}{6}+\frac{\omega}{4}\right)\ln\omega-\left(\frac{13}{60}\omega^3-\frac{\omega^4}{40}-\frac{8}{15}\omega^2+\frac{4\omega}{15}\right)\frac{2t(\omega)}{(4\omega-\omega^2)^{1/2}},$$

$$J_{31}=\frac{-1}{4M^2M_p|\mathbf{p}|}\left[2\omega_<\ln\frac{1-\beta}{1+\beta}+\tfrac{1}{2}\ln^2\frac{1-\beta}{1+\beta}-2\pi^2-2L\left(\frac{2\beta}{1+\beta}\right)\right],$$

where $\mathbf{p}$ is the three momentum of the electron and

$$\beta=|\mathbf{p}|/E_e,$$

$$J_{32}=\frac{P}{2M^2M_p{}^2}\left[\frac{\omega-2}{2}\ln\omega+2\ln\frac{2p_e\cdot P}{M_p{}^2}+1+(4\omega-\omega^2)^{1/2}t(\omega)\right]+\frac{p_e}{2M^2M_p|\mathbf{p}|}\ln\frac{1-\beta}{1+\beta},$$

$$J_{33}=\frac{1}{M_p{}^2}\left[\frac{1}{3}+\frac{3-\omega}{6}\ln\omega+\frac{4-5\omega+\omega^2}{3(4\omega-\omega^2)^{1/2}}t(\omega)\right],$$

$$J_{34}=\frac{P}{M_p{}^2}[\tfrac{5}{8}-\tfrac{1}{4}\omega+(\tfrac{1}{4}-\tfrac{1}{2}\omega+\tfrac{1}{8}\omega^2)\ln\omega$$
$$+(-\omega+\tfrac{3}{4}\omega^2-\tfrac{1}{8}\omega^3)2t(\omega)/(4\omega-\omega^2)^{1/2}],$$

$$J_{35}=\frac{1}{6}-\frac{7}{10}\omega+\frac{\omega^2}{5}+\left(-\frac{\omega}{2}+\frac{\omega^2}{2}-\frac{\omega^3}{10}\right)\ln\omega$$
$$+\left(\frac{\omega^4}{10}-\frac{7}{10}\omega^3+\frac{13}{10}\omega^2-\frac{2}{5}\omega\right)\frac{2t(\omega)}{(4\omega-\omega^2)^{1/2}}.$$

We may now combine these contributions and we have for the total virtual photon matrix element in this formalism

$$\Delta M_V{}^{\text{Stück.}}=\sum_{i=1}^{13}M_i.$$

When this is done we see that this matrix element is independent of the ultraviolet cutoff. This is of course only after we have performed mass and charge renormalizations. Nevertheless the result is somewhat surprising since this is *not* the case for the calculations using the conventional theory.[6] In its place, however, we have a dependence on the mesons' mass M for which we may not take the limit $M\to\infty$. Consequently the limit $M\to\infty$ does not yield the answer obtained using the point interaction.[25] In a sense therefore, the boson mass M has provided a cutoff which renders the over-all answer finite. It may be that by a suitable choice of gauge for the photon propagator one can carry through the whole calculation so that the cutoff appears only in the "charge renormalization" diagrams.[26] It is not obvious that this can be done because we have *four* diagrams which are at present cutoff-dependent.

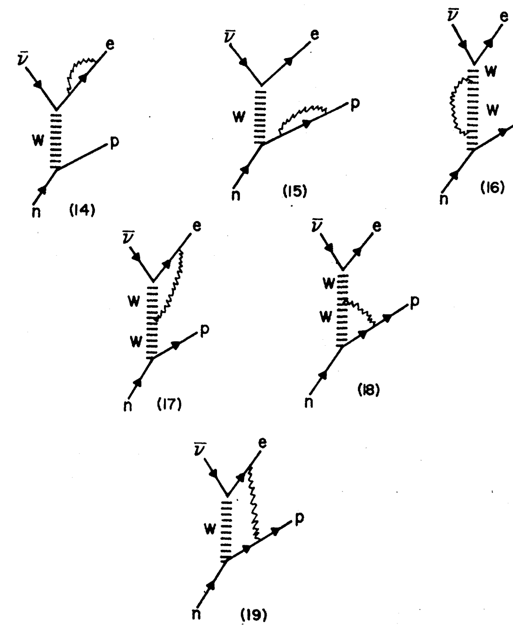

FIG. 14. Virtual photon diagrams: Proca theory.

Finally, after performing spin sums and integrating over phase space, we have for the virtual photon contribution to the transition probability in this formalism

$$P_V(E_e)dE_e=\frac{f^4}{4\pi^3M^4}E_e{}^2(E_e-E_0)^2dE_e\left\{1-\frac{\alpha}{2\pi}\left[4\omega_<+\frac{2}{\beta}\omega_<\ln\frac{1-\beta}{1+\beta}-\ln\frac{M_p}{m_e}+\frac{9}{2}+\frac{1}{2\beta}\ln^2\frac{1-\beta}{1+\beta}\right.\right.$$
$$\left.\left.-\frac{2}{\beta}\left(\pi^2+L\left(\frac{2\beta}{1+\beta}\right)\right)-\left(\beta+\frac{1}{\beta}\right)\ln\frac{1+\beta}{1-\beta}+2\ln\frac{2p_e\cdot P}{M_p{}^2}+1-\frac{3\omega}{2}\ln\omega+\frac{3\omega^2-12\omega}{(4\omega-\omega^2)^{1/2}}t(\omega)\right]\right\},$$

where

$$E_0=M_n-M_p.$$

(b) Proca Formalism

The six diagrams which contribute when one uses the Proca formalism are shown in Fig. 14. In fact we only need to calculate one diagram: (18).

$$M_{14}+M_{15}=\theta[\ln(\lambda^2/m_eM_p)-2\ln(m_eM_p/\lambda_{\min}{}^2)+3](\bar{e}\gamma^\alpha a\nu_1)(\bar{p}\gamma_\alpha a''n),\quad M_{16}=M_{17}=0,$$
$$M_{18}=-\theta\{(\bar{e}\gamma^\alpha a\nu_1)(\bar{p}\gamma_\alpha a''n)[\tfrac{3}{2}\ln(\lambda^2/M^2)-1-\tfrac{1}{2}\omega+(\tfrac{1}{4}\omega^2-\omega+\tfrac{1}{2})\ln\omega+(\tfrac{1}{2}\omega-1)(4\omega-\omega^2)^{1/2}t(\omega)]$$
$$+[\bar{e}(\mathbf{P}/M_p)a\nu_1](\bar{p}a''n)[\tfrac{2}{3}\omega+(-\tfrac{1}{3}\omega^2+\tfrac{1}{2}\omega+1)\ln\omega-\tfrac{1}{3}(2\omega+1)(4\omega-\omega^2)^{1/2}t(\omega)]\}.$$

As before, we may set $M_{19}=M_{13}$ so long as we replace $\ln(\lambda^2/M^2)$ by $\ln(\lambda^2/M^2)-\frac{1}{2}$. Then combining these matrix elements we obtain

$$\Delta M_V{}^{\text{Proca}}=\sum_{i=14}^{19}M_i$$

[25] See S. M. Berman and A. Sirlin, Ann. Phys. **20**, 20 (1962).
[26] D. B. Pearson and J. C. Taylor, Nucl. Phys. **37**, 689 (1962).

$$\Delta M_V{}^{\text{Proca}}-\Delta M_V{}^{\text{Stück.}}=-\theta\{(\bar{e}\gamma^\alpha a\nu_1)(\bar{p}\gamma_\alpha an)[\tfrac{3}{2}\ln(\lambda^2/M^2)-\tfrac{1}{2}\omega+(\tfrac{1}{4}\omega^2-\tfrac{3}{2}\omega+\tfrac{3}{2})\ln\omega+(\tfrac{1}{2}\omega-2)(4\omega-\omega^2)^{1/2}t(\omega)]$$
$$+[\bar{e}(\boldsymbol{P}/M_p)a\nu_1](\bar{p}a''n)[-1+\tfrac{2}{3}\omega+(\omega-\tfrac{1}{3}\omega^2)\ln\omega+\tfrac{2}{3}(1-\omega)(4\omega-\omega^2)^{1/2}t(\omega)]\}.$$

It should be noticed now that although this matrix element is not cutoff-independent the cutoff enters with exactly the same coefficient and sign as in muon decay. This means that we can drop this term in *both* processes and regard it, along with other terms like this, as a renormalization of the coupling constant. In this way the cutoff has dropped out of the Proca formalism in just the same way as it was removed from the Stückelberg formalism. With this renormalization we shall have for muon decay

$$\Delta M_V{}^{\text{Proca}}=\Delta M_V{}^{\text{Stück.}},$$

and for neutron decay

$$\Delta M_V{}^{\text{Proca}}-\Delta M_V{}^{\text{Stück.}}=-\theta\{(\bar{e}\gamma^\alpha a\nu_1)(\bar{p}\gamma_\alpha a''n)[9/4-\tfrac{1}{2}\omega+(\tfrac{1}{4}\omega^2-\tfrac{3}{2}\omega+\tfrac{3}{2})\ln\omega+(\tfrac{1}{2}\omega-2)(4\omega-\omega^2)^{1/2}t(\omega)]$$
$$+[e(\boldsymbol{P}/M_p)a\nu_1](\bar{p}an)[-1+\tfrac{2}{3}\omega+(\omega-\tfrac{1}{3}\omega^2)\ln\omega+\tfrac{2}{3}(1-\omega)(4\omega-\omega^2)^{1/2}t(\omega)]\}.$$

Finally, for the virtual photon transition probability we have

$$P_V{}^{\text{Proca}}(E_e)dE_e=P_V{}^{\text{Stück.}}(E_e)dE_e+(f^4/4\pi^3M^4)E_e{}^2(E_e-E_0)^2dE_e(\alpha/2\pi)$$
$$\times[2-\tfrac{1}{3}\omega+(\tfrac{1}{6}\omega^2-5\omega/4+\tfrac{3}{2})\ln\omega+(\tfrac{1}{3}\omega-11/6)(4\omega-\omega^2)^{1/2}t(\omega)].$$

(c) Bremsstrahlung Diagrams

The eight inner bremsstrahlung diagrams which contribute in the Stückelberg formalism are shown in Fig. 15. However, to our approximation the total bremsstrahlung matrix element,

$$M_{IB}=\sum_{i=20}^{24}M_i,$$

is just the same as for the case *without* vector bosons. Similar considerations apply to the diagrams in the Proca theory; these are shown in Fig. 16. Then the bremsstrahlung contribution to the transiton probability in either formalism is

$$P_{IB}(E_e)dE_e=\frac{\alpha f^4}{8\pi^4M^4}E_e{}^2(E_e-E_0)^2dE_e\left\{4\omega_<-4\ln\frac{2(E_0-E_e)}{m_e}+\frac{28}{3}-\frac{4}{3}\frac{E_0}{E_e}+\frac{E_0{}^2+6E_eE_0-31E_e{}^2}{12\beta E_e{}^2}\ln\frac{1+\beta}{1-\beta}+\right.$$
$$\left.\frac{1}{\beta}\times\left[2\omega_<\ln\frac{1-\beta}{1+\beta}-\ln\frac{1-\beta}{1+\beta}\ln\frac{2(E_0-E_e)^2}{E_em_e}+L\left(\frac{1-\beta}{2}\right)-L\left(\frac{1+\beta}{2}\right)+2L(\beta)-2L(-\beta)\right]\right\}.$$

We now combine the contributions of the virtual and bremsstrahlung photons. Following Kinoshita and Sirlin[6] we make the approximation of neglecting the electron's mass wherever this does not lead to a spurious divergence. This means we set $\beta=1$; then writing $x=E_e/E_0$, we have for the Stückelberg formalism

$$P^{\text{Stück.}}(x)dx=\frac{f^4}{4\pi^3M^4}E_0{}^5(1-x)^2x^2dx\left\{1+\frac{\alpha}{2\pi}\left[3\ln\frac{M_p}{2E_0}-\frac{7}{2}+2\pi^2-\frac{2\pi^2}{3}+4(\ln x-1)\left(\frac{1-x}{3x}-\frac{3}{2}+\ln\frac{1-x}{x}\right)\right.\right.$$
$$\left.\left.+\frac{(1-x)^2}{6x^2}\ln x+\Omega\left(\frac{4(1-x)}{3x}-3+\frac{(1-x)^2}{6x^2}+4\ln\frac{1-x}{x}\right)+\frac{3\omega}{2}\ln\omega+\frac{12\omega-3\omega^2}{(4\omega-\omega^2)^{1/2}}t(\omega)\right]\right\},$$

where $\Omega=\ln(2E_0/m_e)$.

In the above expression the $2\pi^2$ term is just the Coulomb contribution given by the F function.[27]

Since this term is included in the ft values it will henceforth be omitted. Integration gives for the lifetime

$$(\tau^{-1})_{\text{Stück.}}=\tau_0{}^{-1}\{1+(\alpha/2\pi)[3\ln(M_p/2E_0)-7.81$$
$$+\tfrac{3}{2}\omega\ln\omega+3(4\omega-\omega^2)^{1/2}t(\omega)]\}.$$

[27] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), p. 680.

For the Proca formalism one has

$$(\tau^{-1})_{\text{Proca}}=\tau_0{}^{-1}\{1+(\alpha/2\pi)[3\ln(M_p/2E_0)-5.81-\tfrac{1}{3}\omega$$
$$+(\tfrac{1}{6}\omega^2+\tfrac{1}{4}\omega+\tfrac{3}{2})\ln\omega+(\tfrac{1}{3}\omega+7/6)(4\omega-\omega^2)^{1/2}t(\omega)]\}.$$

Just as for the muon decay case the lifetime again satisfies Kinoshita's theorem.[18] But this was obvious earlier since the dependence of the transition probability upon Ω is identical to that obtained using the conventional theory. As remarked before we may not take the

TABLE I. The muon lifetime discrepancy.

	No vector mesons	Stückelberg $M=1.3M_p$	Stückelberg $M=M_K$	Proca $M=1.3M_p$	Proca $M=M_K$
No Corrections	1.65%	1.65	1.65	1.65	1.65
Muon decay radiative corrections	0.42%	0.42	0.42	0.42	0.42
O^{14} radiative corrections	1.7%	1.78	1.43	1.69	1.2
Nonradiative: screening	0.265%	0.265	0.265	0.265	0.265
Nonradiative: vector bosons	0	−0.45	−2.75	−0.45	−2.75
Nuclear physics	−1.17%	−1.17	−1.17	−1.17	−1.17
Discrepancy	+2.865%	+2.50	−0.155	+2.40	−0.385

limit $M\to\infty$. However, for large values of ω the radiative corrections behave like

$$6\ln(M/M_p),$$

which should be compared with $6\ln(\lambda/M_p)$ obtained previously. So that the boson mass is indeed an effective cutoff. This result was also obtained by Lee using the ξ-limiting formalism.[28]

Now we are ultimately interested in the beta decay of O^{14}, since this is the process used to determine the coupling constant. To obtain the radiative corrections for this process one simply inserts the relevant value of E_0. Bardin *et al.*[29] have calculated E_0 for O^{14}

$$E_0{}^{O^{14}}=1.8126\pm0.0014\text{ MeV}.$$

Using this number we can calculate the electromagnetic corrections to the lifetime for three values of ω. We take $\omega=0.277$ which is the smallest possible value, $\omega=1$ corresponding to $M=M_p$, and $\omega=1.7$ which corresponds to $M=1.3M_p$.

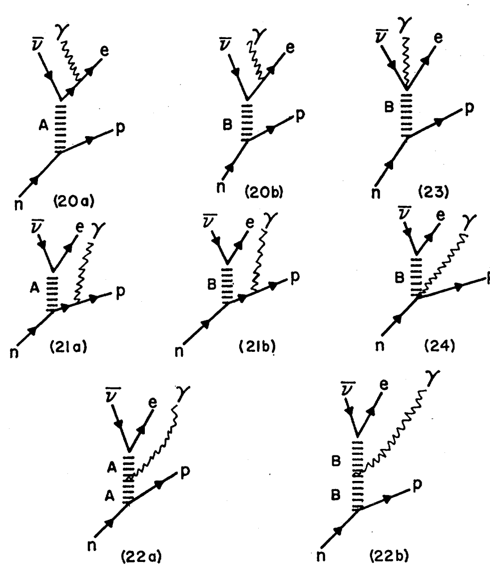

FIG. 15. Bremsstrahlung diagrams in Stückelberg theory.

Then for the Stückelberg formalisms

$$\omega=1.3\;:\;\Delta\tau/\tau_0=-1.78\%,$$
$$\omega=1.0\;:\;\Delta\tau/\tau_0=-1.66\%,$$
$$\omega=0.277:\;\Delta\tau/\tau_0=-1.43\%,$$

and for the Proca formalism

$$\omega=1.3\;:\;\Delta\tau/\tau_0=-1.69\%,$$
$$\omega=1.0\;:\;\Delta\tau/\tau_0=-1.54\%,$$
$$\omega=0.277:\;\Delta\tau/\tau_0=-1.20\%.$$

These values should be compared with the value $\Delta\tau/\tau_0=-1.7\%$ obtained using the conventional theory.

5. THE MUON LIFETIME DISCREPANCY

The situation with regard to the muon lifetime is somewhat confused. In the first place the experimental value is uncertain seeing as how the latest measurement by Farley[30] has not yet been published. We base our conclusions on the previous value obtained by Charpak *et al.*[31] The corrections to the nuclear matrix element are not completely decided[23,24]; we use the later value of

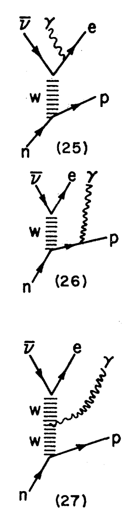

FIG. 16. Bremsstrahlung diagrams in Proca theory.

[28] T. D. Lee, Phys. Rev. **128**, 899 (1962).

[29] R. K. Bardin, C. A. Barnes, W. A. Fowler, and P. A. Seeger, Phys. Rev. **127**, 583 (1962).

[30] F. J. M. Farley, T. Massam, T. Muller, and A. Zichichi, *Proceedings of the 1962 Conference on High Energy Physics at CERN* (CERN, Geneva, 1962), p. 415.

[31] A. Charpak, F. J. M. Farley, R. L. Farwin, T. Muller, J. Sens, and A. Zichichi, Phys. Letters **1**, 16 (1962).

Weidenmuller.[24] Nonradiative effects due to electron screening, nuclear electromagnetic form factor effects, K-capture competition and second forbidden matrix element corrections seem to be agreed upon.[32] All of these effects are included in Table I.

It is fair to say that a vector boson *can* explain the current muon lifetime discrepancy, although its mass should not be much greater than 500 MeV. If the indications from CERN[3] that $M \approx 1.3M_p$ are confirmed, we can only conclude that a sizeable discrepancy still remains to be explained.

[32] L. Durand, L. F. Landowitz, and R. B. Marr, Phys. Rev. Letters 4, 620 (1960).

ACKNOWLEDGMENTS

It is a pleasure to thank Dr. J. C. Taylor who suggested this problem and who provided much advice and encouragement during the course of the work. I am also grateful to Professor S. B. Treiman for some helpful suggestions. In addition, I am indebted to DSIR for a research maintenance grant, and to Princeton University for a Jane Eliza Procter Visiting Fellowship.

PHYSICAL REVIEW VOLUME 135, NUMBER 1B 13 JULY 1964

Diffraction Width in Terms of f^0 and the Residue of the Pomeranchuk Pole*

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At a sharp resonance the phase of a Regge residue $\beta(t)$ should be essentially a multiple of 2π. The value it takes determines to a large extent the falloff rate of β and of the reduced residue $\gamma=\beta/\nu^\alpha$ for $t\leqslant 0$. If f^0 lies on the Pomeranchuk trajectory and if the phase there is 2π then it turns out that $\gamma(t)$ falls off exponentially for small $-t$ with a width comparable to the one deduced from the widths of the high-energy diffraction peaks, and for large $-t$, $\gamma(t)$ has a power fall-off. On the other hand, if the phase at f^0 remains small then the width is at least an order of magnitude larger. The latter case is indicated on the basis of the potential theory results. However, it is possible that the former may be a purely relativistic phenomenon peculiar to the Pomeranchuk pole in which case the Regge-pole hypothesis would be consistent with the high-energy experiments.

UNITARITY implies that a Regge-pole term

$$\beta(t)/l-\alpha(t) \tag{1}$$

near a sharp resonance at $t=t_0$, $\alpha_R(t_0)=l$ satisfies the relations $\beta_R\approx\Gamma$ (the width of the resonance) and $\beta_I\ll\beta_R$. Since β_R is positive at t_0, the latter condition implies that the phase of β must essentially be a multiple of 2π at a resonance.[1] This is a strong restriction on the phase; the value it takes, namely, ≈ 0, 2π, etc., determines to a large extent how fast β or rather the reduced residue γ (see below) falls off for $t<0$. The behavior in the negative t region is of some interest since it was pointed out recently[2] that if γ of the Pomeranchuk pole (P) showed a sharp diffraction-type fall off for small $-t$, then the Regge-pole approximation[3] may still be adequate in explaining the high-energy behavior of scattering amplitudes. The question of shrinkage or absence of it can then be understood in terms of appropriate linear combinations of P with other important poles.[2] We shall show below that if f^0 lies on the P trajectory and if the phase is 2π at t_f then one obtains an exponential type falloff for small $-t$, with a width comparable to the one observed experimentally, and a power falloff for large $-t$.[4] Potential theory results, however, indicate that near a resonance the phase should stay small and not approach 2π.[5] If this is assumed to be true also for P then we find that it is impossible for γ to achieve a diffraction-type behavior; the width turns out to be at least an order of magnitude larger than the experimental values. This would strongly suggest that the pole-hypothesis is inadequate and that perhaps other singularities in addition to the commonly assumed poles play an important role.

Consider elastic $\pi\pi$ scattering with s the square of the c.m. energy and t the square of the momentum transfer. We shall take BeV as the unit of mass. For large s, the contribution of a Regge pole with position $\alpha(t)$, to the scattering amplitude $A_I(s,t)$ is given by

$$A_I(s,t)=-\rho_I\frac{2^{\alpha-1}(\pi)^{1/2}(2\alpha+1)\Gamma(\frac{1}{2}+\alpha)}{\Gamma(1+\alpha)}\times\left(\frac{1+e^{-i\pi\alpha}}{\sin\pi\alpha}\right)\gamma(t)\left(\frac{s}{2M^2}\right)^\alpha, \tag{2}$$

* Work supported in part by the U. S. Atomic Energy Commission.

[1] More precisely, $2n\pi+O(\alpha_I)$. For a sharp resonance, $\alpha_I\ll 1$.

[2] B. R. Desai, Phys. Rev. Letters 11, 59 (1963).

[3] References to the earlier theoretical work are given in Ref. 2.

[4] Even if f^0 turns out to be 1^- or 3^- [see W. Frazer. S. Patil, and N. Xuong (to be published)] the essential points of this paper will not change provided that there exists a 2^+ particle on the P trajectory at a higher mass ($\lesssim 2$ BeV).

[5] A. Ahmadzadeh, thesis, University of California (unpublished). The results, based on Yukawa potentials, were communicated to me by G. F. Chew.